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Dual self-transform function

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Abstract. The Dirac comb is quoted as the only self-Fourier–Fresnel function. Other dual self-transform functions (DSTFs) known to us are approximate. We show that for arbitrary transformable even periodic function $p(x, y)$, the functions $f(x, y) = p(x, y)\mathrm{comb}(x, y) + p^{\mathrm{Fo}}(x, y) * \mathrm{comb}(x, y)$ are DSTFs.

Some functions are both their own Fourier transform and their own Fresnel transform. The Dirac comb function, which has been widely used, is an example [1]. Call them dual self-transform functions (DSTF). Only some approximate DSTFs

$$f(x, y) = [c(x, y) * \mathrm{comb}(x, y)]a(x, y) \tag{1}$$

and

$$f(x, y) = [a(x, y)\mathrm{comb}(x, y)] * c(x, y) \tag{2}$$

were given in literature [1], where $*$ is a convolution operator, $c(x, y)$ is an even cell function, $a(x, y)$ is an even aperture function and

$$\begin{aligned}\mathrm{comb}(x, y) &= \sum_{j,k} \delta(x - j, y - k) = \sum_{j,k} \delta(x - j)\delta(y - k)\\ &= \sum_{j} \delta(x - j) \sum_{k} \delta(y - k) = \mathrm{comb}(x)\mathrm{comb}(y) \qquad j, k = \ldots, -1, 0, 1, \ldots.\end{aligned}$$

The reasons why DSTFs are approximate has been explained in the literature and it has given the impression that the Dirac comb function is the only DSTF and it is unfeasible to assemble an exact one. We wish to point out that there is an infinity of exact DSTF: given an arbitrary two-dimensional transformable even periodic function $p(x, y)$ whose periods along the x and y axes are both rational numbers, then the function

$$f(x, y) = p(x, y)\mathrm{comb}(x, y) + p^{\mathrm{Fo}}(x, y) * \mathrm{comb}(x, y) \tag{3}$$

is a DSTF, where the superscript, Fo, expresses the Fourier transform, i.e.

$$p^{\mathrm{Fo}}(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(s, t) \exp[-\mathrm{i}2\pi(sx + ty)]\mathrm{d}s\, \mathrm{d}t.$$

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Proof.

(1) The Fourier transform of $f(x, y)$ is itself.

$$
\begin{aligned}
f^{\mathrm{Fo}}(x, y) &= [p(x, y)\mathrm{comb}(x, y) + p^{\mathrm{Fo}}(x, y) * \mathrm{comb}(x, y)]^{\mathrm{Fo}} \\
&= p^{\mathrm{Fo}}(x, y) * \mathrm{comb}(x, y) + p(-x, -y)\mathrm{comb}(x, y).
\end{aligned}
$$

Notice that $p(x, y)$ is even

$$p(-x, -y) = p(x, y)$$

then $f^{\mathrm{Fo}}(x, y) = f(x, y)$.

(2) $f(x, y)$ is its own Fresnel transform. □

A two-dimensional Fresnel transform is defined by

$$[f(x, y)]^{\mathrm{Fr}} = \frac{\mathrm{i}}{\lambda z}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(s, t)\exp\left\{\mathrm{i}\frac{\pi}{\lambda z}[(s-x)^2 + (t-y)^2]\right\}\mathrm{d}s\,\mathrm{d}t \tag{4}$$

where λ and z usually express wavelength and distance respectively in optics. This integral is used to describe Fresnel diffraction of light waves in free space through distance z. $f(x, y)$ is the wavefunction. If $f(x, y)$ is its own Fresnel transform for some particular values of z, the wavefunction will reproduce itself at these distances. This effect is called self-imaging or the fractional Talbot effect [2, 3] and then this wavefunction, $f(x, y)$, is called a self-Fresnel function [1]. Assume that the periods of function $p(x, y)$ are M_x/N_x and M_y/N_y along the x and y axes respectively (M_x, N_x, M_y and N_y are all integers), and expand $p(x, y)$ into its Fourier series, then equation (3) can be reformulated as

$$
\begin{aligned}
f(x, y) &= \left\{\left[\left(\sum_{m,n} C_{m,n}\exp\left[\mathrm{i}2\pi\left(m\frac{N_x}{M_x}x + n\frac{N_y}{M_y}y\right)\right]\right)\mathrm{comb}(x, y)\right]^{\mathrm{Fo}}\right\}^{\overline{\mathrm{Fo}}} \\
&\quad + \{[p^{\mathrm{Fo}}(x, y) * \mathrm{comb}(x, y)]^{\mathrm{Fo}}\}^{\overline{\mathrm{Fo}}} \\
&= \left[\sum_{m,n} C_{m,n}\sum_{j,k}\delta\left(u - j - m\frac{N_x}{M_x}, v - k - n\frac{N_y}{M_y}\right) + p(u, v)\mathrm{comb}(u, v)\right]^{\overline{\mathrm{Fo}}}
\end{aligned} \tag{5}
$$

where $C_{m,n}$ is the coefficient of the Fourier series, and $\overline{\mathrm{Fo}}$ indicates the inverse Fourier transform. In equation (5) we have used the following combinations:

$$\mathrm{comb}(x, y)^{\mathrm{Fo}} = \mathrm{comb}(u, v)$$

and

$$
\begin{aligned}
&\left[\left(\sum_{m,n} C_{m,n}\exp\left[\mathrm{i}2\pi\left(m\frac{N_x}{M_x}x + n\frac{N_y}{M_y}y\right)\right]\right)\mathrm{comb}(x, y)\right]^{\mathrm{Fo}} \\
&\qquad = \sum_{m,n} C_{m,n}\left\{\exp\left[\mathrm{i}2\pi\left(m\frac{N_x}{M_x}x + n\frac{N_y}{M_y}y\right)\right]\right\}^{\mathrm{Fo}} * [\mathrm{comb}(x, y)]^{\mathrm{Fo}} \\
&\qquad = \sum_{m,n} C_{m,n}\delta\left(u - m\frac{N_x}{M_x}, v - n\frac{N_y}{M_y}\right) * \mathrm{comb}(u, v) \\
&\qquad = \sum_{m,n} C_{m,n}\delta\left(u - m\frac{N_x}{M_x}, v - n\frac{N_y}{M_y}\right) * \sum_{j,k}\delta(u - j, v - k)
\end{aligned}
$$

$$= \sum_{m,n} C_{m,n} \sum_{j,k} \delta\left(u - m\frac{N_x}{M_x}, v - n\frac{N_y}{M_y}\right) * \delta(u-j, v-k)$$

$$= \sum_{m,n} C_{m,n} \sum_{j,k} \delta\left(u - j - m\frac{N_x}{M_x}, v - k - n\frac{N_y}{M_y}\right).$$

We rewrite equation (4) in the form of

$$[f(x,y)]^{\mathrm{Fr}} = f(x,y) * \left\{\frac{\mathrm{i}}{\lambda z}\exp\left[\mathrm{i}\frac{\pi}{\lambda z}(x^2+y^2)\right]\right\} = \{[f(x,y)]^{\mathrm{Fo}}\exp[-\mathrm{i}\pi\lambda z(u^2+v^2)]\}^{\overline{\mathrm{Fo}}}. \tag{6}$$

By putting $f(x)$ (given by equation (5)) into equation (6), we obtain

$$\begin{aligned}[f(x,y)]^{\mathrm{Fr}} = \Bigg\{ & \sum_{m,n} C_{m,n} \sum_{j,k} \delta\left(u - j - m\frac{N_x}{M_x}, v - k - n\frac{N_y}{M_y}\right) \\ & \times \exp\left[-\mathrm{i}\pi\lambda z\left[\left(m\frac{N_x}{M_x}+j\right)^2 + \left(n\frac{N_y}{M_y}+k\right)^2\right]\right] \\ & + p(u,v)\sum_{j,k}\delta(u-j, v-k)\exp[-\mathrm{i}\pi\lambda z(j^2+k^2)]\Bigg\}^{\overline{\mathrm{Fo}}}.\end{aligned}$$

Because M_x, M_y, N_x and N_y are all integers, it is clear than when

$$z = \frac{2K}{\lambda}M_x^2 M_y^2 \qquad K = 1, 2, \cdots$$

$$\exp\left[-\mathrm{i}\pi\lambda z\left[\left(m\frac{N_x}{M_x}+j\right)^2 + \left(n\frac{N_y}{M_y}+k\right)^2\right]\right] = 1$$

and

$$\exp[-\mathrm{i}\pi\lambda z(j^2+k^2)] = 1 \qquad \text{for } m, n, j, k = 0, \pm 1, \pm 2, \ldots$$

we have

$$f^{\mathrm{Fr}}(x,y) = f(x,y)$$

i.e. $f(x,y)$ defined by equation (3) are dual self-transform functions.

Example. Let

$$p(x,y) = \left[\Lambda(x) * \sum_m \delta(x-4m)\right]\left[\Lambda(y) * \sum_n \delta(y-4n)\right] \qquad m, n = 0, \pm 1, \pm 2, \ldots \tag{7}$$

and

$$\Lambda(t) = \begin{cases} 1 + t/2 & -2 \leqslant t \leqslant 0 \\ 1 - t/2 & 0 \leqslant t \leqslant 2 \\ 0 & t = \text{others} \end{cases} \tag{8}$$

where $\Lambda(t)$ is the tower function and the function $p(x,y)$ is an even periodic function. By substituting $p(x,y)$ into equation (3), we have the dual self-transform function

$$f(x,y) = \sum_m [\tfrac{1}{2}\delta(x+1-4m) + \delta(x-4m) + \tfrac{1}{2}\delta(x-1-4m)]$$

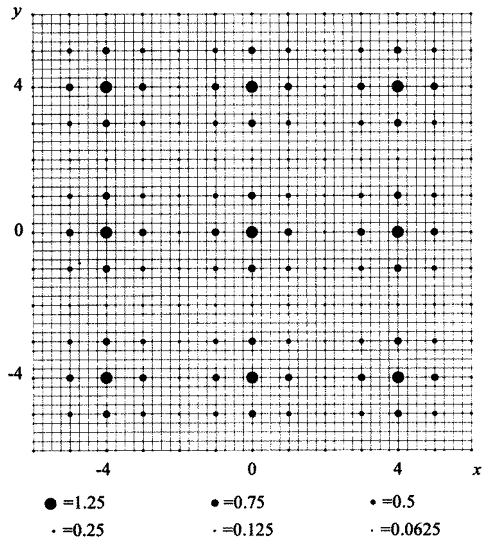

Figure 1. An example of the dual self-transform function. Every black dot is a $\delta(x, y)$ function, and its size expresses the intensity of the $\delta(x, y)$.

$$\times \sum_n \left[\tfrac{1}{2}\delta(y+1-4n) + \delta(y-4n) + \tfrac{1}{2}\delta(y-1-4n) \right]$$
$$+\frac{1}{4}\sum_j \left[\delta\left(x-\frac{j}{4}\right)\left(1+\cos\frac{j}{2}\pi\right) \right] \times \frac{1}{4}\sum_k \left[\delta\left(y-\frac{k}{4}\right)\left(1+\cos\frac{k}{2}\pi\right) \right] \tag{9}$$

which is illustrated in figure 1.

The application of self-Fourier function is the design of the laser resonator cavities [4, 5]. The self-Fresnel function can self-image in free space, and this effect of a grating has many applications [2, 3, 6]. DSTF has either property of them, i.e. it can be used as self-Fourier function or self-Fresnel function. In addition, it has both properties of them. Using it to design a laser resonator cavity, we can get a laser whose output wavefront can self-image in free space at some particular distance or on the Fourier plane of a lens. The other application is being considered for developing a multiple-channel optical system [1].

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